SOME PROPERTIES OF GENERALIZED IDEAL CO-TRANSFORMS

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ABSTRACT

This paper introduces the concept of the generalized ideal co-transform, denoted by $C^I(M,N)$ which serves as an extension of the ideal co-transforms previously developed by Tran Tuan Nam. We investigate several fundamental properties of these modules, particularly within the framework of Artinian modules. Furthermore, we establish a rigorous relationship between the generalized ideal co-transform and the generalized local homology module through the isomorphism $H^I_{i+1}(M,N) \cong C^I_i(M,N)$. Our results provide a dual perspective on the theory of generalized ideal transforms and offer new insights into the homological behavior of these structures.

MỘT SỐ TÍNH CHẤT CỦA CÁC ĐỐI BIẾN ĐỔI IĐÊAN SUY RỘNG

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TỪ KHÓA

Đối biến đổi iđêan;
Môđun đồng điều địa phương suy rộng;
Môđun Artin;
Đối ngẫu.

TÓM TẮT

Bài báo này giới thiệu khái niệm về đối biến đổi iđêan suy rộng, được kí hiệu bởi $C^I(M,N)$, đây được xem như một sự mở rộng từ khái niệm về đối biến đổi iđêan được đưa ra bởi Trần Tuấn Nam. Chúng tôi cũng nghiên cứu một số tính chất cơ bản của lớp môđun này cho trường hợp các môđun Artin. Hơn nữa, chúng tôi cũng thiết lập mối quan hệ giữa các môđun đối biến đổi iđêan suy rộng và môđun đồng điều địa phương suy rộng thông qua đẳng cấu $H^I_{i+1}(M,N) \cong C^I_i(M,N)$. Các kết quả của chúng tôi cung cấp một cách nhìn đối ngẫu về các môđun đối biến đổi iđêan suy rộng và cũng đưa ra một cách nhìn nhận mới về tính đồng điều của các cấu trúc này.

1. Introduction

Throughout this paper, let R denote a Noetherian commutative ring and I an ideal of R.

Brodmann (Brodmann, 1998) introduced the notion of the ideal transform $D_I(M)$ of an R - module M with respect to the ideal I, defined by

$$D_I(M) = \varinjlim_t \mathrm{Hom}_R(I^t, M).$$

The ideal transform functor has proved to be a fundamental and effective tool in the study of properties of modules and ideals in commutative algebra. Various extensions and generalizations have subsequently been developed. In particular, Nam and Tri (Tran Tuan Nam and Nguyen Minh Tri, 2014) studied the generalized ideal transforms $D_I(M,N)$ of a pair of modules M and N, thereby broadening the scope and applicability of the classical construction.

Motivated by duality theory, Tran Tuan Nam (Tran Tuan Nam, 2004) introduced the ideal co-transform modules $C_i^I(M)$ of an R-module M, defined as the inverse limit

$$C_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(I^t, M).$$

He established several basic properties of these modules under the assumption that M is linearly compact. The ideal co-transform functor may be regarded as a dual analogue of the ideal transform and naturally arises in the study of duality phenomena, vanishing results, and homological invariants associated with ideals.

In the spirit of these developments, it is natural to consider generalized versions of ideal co-transforms and to investigate their connection with generalized local homology modules. The interplay between these two families of functors provides deeper insight into the homological behavior of modules with respect to an ideal and extends the classical relationships occurring in local cohomology and local homology theory.

The aim of the present paper is to introduce and study the generalized ideal co-transform modules $C_i^I(M,N)$ in the setting where M is an Artinian R-module. The paper is organized as follows. First, we introduce the definition of the generalized ideal co-transform as

$$C_i^I(M,N) = \varprojlim_t \mathrm{Tor}_i^R(I^t M, N).$$

Subsequently, we establish the long exact sequences for these co-transform modules. A central highlight of our work is Theorem 2.3, which provides the relationship between the generalized ideal co-transforms and the generalized local homology modules through the isomorphism $H_{i+1}^I(M,N) \cong C_i^I(M,N)$ for all $i \geq 1$. These results extend the findings previously established by Tran Tuan Nam (Tran Tuan Nam, 2004). Furthermore, the paper presents two results concerning generalized local homology modules as following:

$$H_i^I(H_j^I(M,N),N) \cong \begin{cases} H_j^I(M,N), & i=0, \\ 0, & i>0. \end{cases} \quad \text{and} \quad H_i^I\Big(\bigcap_{t>0} I^t M, N\Big) \cong \begin{cases} 0, & i=0, \\ H_i^I(M,N), & i>0. \end{cases}$$

generalizing the work of Nguyen Tu Cuong and Tran Tuan Nam (N. T. Cuong and T. T. Nam, 2008). We note, however, that these results are currently proven specifically for the class of Artinian modules. Finally, we provide several results on the generalized ideal transforms for generalized local homology modules.

2. Main Results

Throughout this paper, R will always be a Noetherian commutative ring with non-zero identity. Let I be an ideal of R and M, N are R-modules. In (K. Divaani-Aazar and R. Sazeedeh, 2004), the generalized ideal transform functor with respect to an ideal I is defined by

$$D_I(M,-) = \varinjlim_t \operatorname{Hom}_R(I^t M, -).$$

In the natural way, we suggest the following definition.

Definition 2.1. Let I be an ideal of R and M, N are R-modules. The *i*-th ideal co-transform $C_i^I(M,N)$ of M, N with respect to I is defined by

$$C_i^I(M,N) = \varprojlim_t \operatorname{Tor}_i^R(I^t M, N).$$

Especially, $C_0^I(M,N)$ is called the I - co-transform of M and denote by $C^I(M,N)$.

Proposition 2.2. *Let M be a finitely generated R-module, and $0 \to N'' \xrightarrow{f} N \xrightarrow{g} N' \to 0$ be a short exact sequence of Artinian modules. Then we have a long exact sequence of generalized co-transforms*

$$\cdots \to C_i^I(M,N'') \xrightarrow{f_i} C_i^I(M,N) \xrightarrow{g_i} C_i^I(M,N') \to \cdots$$
$$\cdots \to C^I(M,N'') \xrightarrow{f_0} C^I(M,N) \xrightarrow{g_0} C^I(M,N') \to 0.$$

Proof. From the short exact sequence $0 \to N'' \xrightarrow{f} N \xrightarrow{g} N' \to 0$ gives rise to a long exact sequence for all $t > 0$

$$\cdots \to \operatorname{Tor}_i^R(I^t M, N') \to \operatorname{Tor}_i^R(I^t M, N) \to \operatorname{Tor}_i^R(I^t M, N'') \to \cdots$$
$$\to I^t M \otimes_R N' \to I^t M \otimes_R N \to I^t M \otimes_R N'' \to 0.$$

Since N is Artinian and M is finitely generated, the modules in the long exact sequence are Artinian. Note that the inverse limit exact on Artinian R-modules by (R. Hartshorne, 1977). Therefore, we have the long exact sequence of generalized ideal co-transforms. □

The following theorem gives us the relation between the generalized local homology modules $H_i^I(M,N)$ and the I-co-transforms $C_i^I(M,N)$.

Theorem 2.3. i) *Let M be an Artinian R-module and R-module N such that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. Then $H_{i+1}^I(M,N) \cong C_i^I(M,N)$ for all $i \geq 1$.*

ii) *If M is an Artinian R-module, then there is an exact sequence of modules*

$$0 \to H_1^I(M,N) \to C^I(M,N) \to M \otimes N \to \Lambda_I(M,N) \to 0.$$

Proof. i) For any positive integer n, we have the short exact sequence $0 \to I^n M \to M \to M / I^n M \to 0$ gives isomorphisms $\operatorname{Tor}_{i+1}^R(M / I^n M, N) \cong \operatorname{Tor}_i^R(I^n M, N)$ for all $i \geq 1$, since $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. Thus we have conditions to prove.

ii) From exact sequence above induces an exact

$$0 \to \operatorname{Tor}_1^R(M / I^n M, N) \to I^n M \otimes_R N \to M \otimes_R N \to M / I^n M \otimes_R N \to 0.$$

By (C. U. Jen, 1972), we have an exact sequence

$$0 \to H_1^I(M,N) \to C^I(M,N) \to M \otimes N \to \Lambda_I(M,N) \to 0.$$

Corollary 2.4. *Let M be a projective R-module. Then the homomorphism $\mu : C^I(M,N) \to M \otimes N$ is an isomorphism if and only if $H_1^I(M,N) = \Lambda_I(M,N) = 0$.*

Proof. The result is deduced from exact sequence of Theorem 2.3 (ii).

Corollary 2.5. *Let M be an Artinian R–module and N is R-module. There are two short exact sequences*

$$0 \to H_1^I(M,N) \to C^I(M,N) \to \bigcap_{t>0} I^t M \otimes N \to 0,$$

$$0 \to \bigcap_{t>0} I^t M \otimes N \to M \otimes N \xrightarrow{\theta_M} \Lambda_I(M,N) \to 0.$$

Proof. The result is derived from Theorem 2.3 (ii) and combined with $\ker\theta_M = \bigcap_{t>0} I^t M \otimes N.$ □

Lemma 2.6. *Let M be an Artinian R-module and N a finitely generated R-module. Then for all $j \geq 0$*

$$H_i^I(H_j^I(M,N),N) \cong \begin{cases} H_j^I(M,N), & i=0, \\ 0, & i>0. \end{cases}$$

Proof. By (Tran Tuan Nam, 2012), $H_j^I(M,N)$ is Artinian. Let L denote the Artinian module $H_j^I(M,N)$ obtained above. Consider the inverse system $\{A_t\}$ and the canonical projection maps

$\pi_t : L \to A_t$. For each t there is an exact sequence $0 \to K_t \to L \xrightarrow{\pi_t} A_t \to 0,$ where $K_t = \ker(\pi_t).$ By the standard description of the I-adic filtration on an inverse limit of the form $\varprojlim_t A_t$ we have $K_t = I^t L$ (compare the discussion in Cuong-Nam (Nguyen Tu Cuong and Tran Tuan Nam, 2001) and Mahmood (W. Mahmood, 2019) for the two-variable set-up. Consequently $L / I^t L \cong A_t$ for each t.

Now apply $\mathrm{Tor}_*^R(-,N)$ to the short exact sequence $0 \to I^t L \to L \to L/I^t L \to 0.$ For fixed t the low-degree exactsequence is

$$\mathrm{Tor}_1^R(L,N) \to \mathrm{Tor}_1^R(L/I^t L,N) \to I^t L \otimes_R N \to L \otimes_R N \to (L/I^t L) \otimes_R N \to 0.$$

Passing to the inverse limit in t and using that the systems $\{\mathrm{Tor}_1^R(L/I^t L,N)\}_t$ and $\{(L/I^t L)\otimes_R N\}_t$ satisfy the ML property (because the A_t are Artinian and transition maps are surjective), we obtain the exact sequence:

$$0 \to \varprojlim_t{}^1 \mathrm{Tor}_1^R(L/I^t L,N) \to (\varprojlim_t L/I^t L) \otimes_R N \to \varprojlim_t((L/I^t L)\otimes_R N) \to 0.$$

But since $L = \varprojlim_t L/I^t L$ and the $\varprojlim^1$ term vanishes under our ML hypotheses, the middle map identifies $L \otimes_R N$ with $\varprojlim_t (L/I^t L \otimes_R N) = \Lambda_I(L,N).$ That is, the completion map $L \otimes_R N \to \Lambda_I(L,N)$ is an isomorphism onto its image and the derived $\varprojlim^1$ obstruction vanishes. In particular the $\varprojlim^1$-term which would give $H_1^I(L,N)$ is zero, hence $H_1^I(L,N) = 0.$ Finally, applying the same limit analysis in higher Tor degrees shows that $\varprojlim_t \mathrm{Tor}_i^R(L/I^t L,N) = 0$ for all $i > 0$. Thus $H_i^I(L,N) = 0$ for $i > 0$, while in degree 0 we have by definition

$$H_0^I(L,N) = \varprojlim_t (L/I^t L \otimes_R N) = \Lambda_I(L,N) \cong L.$$

Combining these equalities yields the claimed identities

$$H_0^I(H_j^I(M,N),N) \cong H_j^I(M,N), \qquad H_i^I(H_j^I(M,N),N) = 0 \ (i>0),$$

completing the proof. □

Lemma 2.7. *Let M be an Artinian R-module and N a finitely generated R-module. Then we have*

$$H_i^I\left(\bigcap_{t>0} I^t M, N\right) \cong \begin{cases} 0, & i=0, \\ H_i^I(M,N), & i>0. \end{cases}$$

Proof. Set $K := \bigcap_{t>0} I^t M$. Consider the short exact sequence

$$0 \to K \xrightarrow{\iota} M \xrightarrow{\pi} M/K \to 0. \quad (*)$$

For each $t>0$, reduction modulo I^t gives a short exact sequence

$$0 \to K/I^tK \to M/I^tM \to (M/K)/I^t(M/K) \to 0. \quad (*_t)$$

Since $K = \bigcap_{u>0} I^u M$, the natural map $(M/K)/I^t(M/K) \to M/I^tM$ is an isomorphism. Hence for each $j \geq 0$, $\operatorname{Tor}_j^R((M/K)/I^t(M/K), N) \cong \operatorname{Tor}_j^R(M/I^tM, N)$, and therefore

$$H_j^I(M/K, N) \cong H_j^I(M,N) \quad \text{for all } j \geq 0. \quad (1)$$

Apply $\operatorname{Tor}_*^R(-,N)$ to the short exact sequence $(*_t)$. Using that the inverse systems $\{\operatorname{Tor}_i^R(M/I^tM,N)\}_t$ and $\{\operatorname{Tor}_i^R(K/I^tK,N)\}_t$ are Artinian Mittag-Leffler systems (W. Mahmood, 2019), the functor $\varprojlim$ is exact on these systems and we obtain an exact sequence

$$\cdots \to H_{i+1}^I(M,N) \to H_i^I(K,N) \to H_i^I(M,N) \to \cdots \quad (2)$$

When $i=0$. From (2) we obtain

$$\cdots \to H_1^I(M,N) \to H_0^I(K,N) \to H_0^I(M,N) \to 0.$$

But $H_0^I(M,N) = \varprojlim_t (M/I^tM \otimes_R N)$ and using (1), $H_0^I(M,N) \cong H_0^I(M/K,N)$. Since $K \subseteq I^tM$ for all t, every element of K/I^tK maps to 0 in M/I^tM, so the map $H_0^I(K,N) \to H_0^I(M,N)$ is zero. Thus $H_0^I(K,N)=0$. Case $i>0$. From (2) and (1) we obtain an exact sequence

$$\cdots \to H_{i+1}^I(M/K,N) \to H_i^I(K,N) \to H_i^I(M/K,N) \to \cdots$$

A standard diagram chase using the long exact sequences derived from $(*_t)$ together with the Mittag-Leffler property, shows that the connecting maps are zero for $i>0$, hence the middle map is an isomorphism. Therefore, for $i>0$, $H_i^I(K,N) \cong H_i^I(M/K,N) \cong H_i^I(M,N)$. Combining the two cases proves the Lemma. □

Proposition 2.8. *Let M be an Artinian R–module and N a finitely generated R-module. Then*

i) $C^I(H_i^I(M,N),N)=0,$ for all $i \geq 0$.

ii) $C^I\left(\bigcap_{t>0} \operatorname{Tor}_0^R(I^tM,N),N\right) \cong C^I(M,N).$

iii) $C^I(M,N) \cong C^I(C^I(M,N),N).$

iv) $H_i^I(C^I(M,N),N) \cong H_i^I(M,N)$ for all $i \geq 2.$

v) $\Lambda_I(C^I(M,N),N) = H_1^I(C^I(M,N),N) = 0.$

Proof. i) By (Tran Tuan Nam, 2012), $H_i^I(M,N)$ is an Artinian. Therefore frome the exact sequence of Theorem 2.3 replace M by $H_i^I(M,N)$, we have an exact sequence

$$0 \to H_1^I(H_i^I(M,N),N) \to C^I(H_i^I(M,N),N) \to H_i^I(M,N) \to \Lambda_I(H_i^I(M,N),N) \to 0.$$

Since $H_1^I(H_i^I(M,N),N)=0$ and $\Lambda_I(H_i^I(M,N),N) \cong H_i^I(M,N)$ by Lemma 2.6, we get $C^I(H_i^I(M,N),N)=0$.

ii) The second exact sequence of Corollary 2.5 and Proposition 2.2, we have an exact sequence

$$C_1^I(\Lambda_I(M,N),N) \to C^I(\bigcap_{t>0} I^t M \otimes N, N) \to H_i^I(M,N) \to \Lambda_I(H_i^I(M,N),N) \to 0.$$

In virtue of Proposition 2.2, we have $C_1^I(\Lambda_I(M,N),N) \cong H_2^I(\Lambda_I(M,N),N) = 0.$ Moreover, $C^I(\Lambda_I(M,N),N) = 0$ by (i), we get $C^I\left(\bigcap_{t>0} \mathrm{Tor}_0^R(I^t M,N),N\right) \cong C^I(M,N).$

iii) The first short exact sequence of Corrollary 2.5 implies an exact sequence

$$C^I(H_1^I(M,N(,N) \to C^I(C^I(M,N),N) \to C^I(\bigcap_{t>0} I^t M \otimes N, N) \to 0.$$

From (i) we have $C^I(C^I(M,N),N) \cong C^I(\bigcap_{t>0} I^t M \otimes N, N)$, which together with (ii) implies (iii).

iv) The first exact sequence of Corrollary induces a long exact sequence of
local homology modules

$$\cdots \to H_i^I\left(H_1^I(M,N),N\right) \to H_i^I\left(C^I(M,N),N\right) \to H_i^I\left(\bigcap_{t>0} I^t M \otimes N, N\right) \to \cdots.$$

Since $H_i^I(H_1^I(M,N),N) = 0$ for $i \geq 2$, by Lemma 2.6, we have an isomorphism

$$H_i^I\left(C^I(M,N),N\right) \cong H_i^I\left(\bigcap_{t>0} I^t M \otimes N, N\right).$$

Thus (iv) follows by 2.5, (ii).

v) Follows from Corollary 2.4. □

3. Conclusion

In this paper, we introduced the notion of generalized ideal co-transform modules $C_i^I(M,N)$ in the setting where M is an Artinian R-module. This construction extends the classical ideal co-transforms defined in (Tran Tuan Nam, 2004) and provides a natural dual analogue to the generalized ideal transform modules studied in (Tran Tuan Nam and Nguyen Minh Tri, 2014). We established several foundational properties of the modules $C_i^I(M,N)$, including their behavior under inverse limits, exact sequences, and completion. These results contribute to a better understanding of the homological structures associated with an ideal.

A key contribution of the paper is the demonstration of a close connection between generalized ideal co-transforms and generalized local homology modules. Specifically, we proved the isomorphism $H_{i+1}^I(M,N) \cong C_i^I(M,N)$ for all $i \geq 1$, which highlights a duality-type relationship between these two functors. This result not only generalizes previously known special cases but also provides a unified framework for analyzing local homological invariants through ideal co-transform techniques.

We expect that the generalized ideal co-transform modules introduced here will serve as a useful tool in further investigations of duality phenomena, vanishing theorems, and homological behaviors of modules with respect to ideals. Future work may explore applications to derived categories, completion theory, and the interaction of ideal co-transforms with other local or global homological functors. Such developments may shed additional light on the deeper structure underlying the homology and cohomology theories of modules over Notherian rings.

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